EXPANSION OF AN ARBITRARY FUNCTION INTO AN INTEGRAL IN TERMS OF ASSOCIATED SPHERICAL FUNCTIONS

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We obtain a theorem for expansion of an arbitrary function into an integral in terms of associated spherical functions, which finds use in solving the boundary value problems of the mathematical physics and the theory of elasticity, for hyperboloids of revolution of one sheet.

1. Introduction. Let us consider the integral expansions of the form

$$f(x)=\frac{4}{\pi}\sum_{n=0}^{[{}^1/_2(m-1)]}(m-{}^1/_2-2n)\,\Gamma(2m-2n)\,\Gamma(2n+1)\,\varphi^m_{m-{}^1/_2-2n}(x)\times$$
$$\times\int_0^\infty f(y)\,\varphi^m_{m-{}^1/_2-2n}(y)\,dy+\frac{2}{\pi}\int_0^\infty\tau\,\mathrm{th}\,\pi\tau\,\Gamma({}^1/_2+m+i\tau)\,\Gamma({}^1/_2+m-i\tau)\times$$
$$\times\,\varphi_{i\tau}{}^m(x)\,d\tau\int_0^\infty f(y)\,\varphi_{i\tau}{}^m(y)\,dy\qquad(0<x<\infty;\ m=0,1,2,\ldots)\qquad(1.1)$$

$$f(x)=\frac{4}{\pi}\sum_{n=0}^{[{}^1/_2m]-1}(m-{}^3/_2-2n)\,\Gamma(2m-2n-1)\,\Gamma(2n+2)\,\psi^m_{m-{}^3/_2-2n}(x)\times$$
$$\times\int_0^\infty f(y)\,\psi^m_{m-{}^3/_2-2n}(y)\,dy+\frac{2}{\pi}\int_0^\infty\tau\,\mathrm{th}\,\pi\tau\,\Gamma({}^1/_2+m+i\tau)\,\Gamma({}^1/_2+m-i\tau)\times$$
$$\times\,\psi_{i\tau}{}^m(x)\,d\tau\int_0^\infty f(y)\,\psi_{i\tau}{}^m(y)\,dy\qquad(0<x<\infty;\ m=0,1,2,\ldots)\qquad(1.2)$$

where $\phi_\nu^m(x)$ and $\psi_\nu^m(x)$ denote, respectively, the even and odd combination of the spherical functions with imaginary arguments

$$\varphi_\nu{}^m(x)=\frac{1}{2}\,[e^{\mp{}^1/_2 i\pi m}P_{\nu-{}^1/_2}^{-m}(ix)+e^{\pm{}^1/_2 i\pi m}P_{\nu-{}^1/_2}^{-m}(-ix)]\qquad(x\gtrless 0)\qquad(1.3)$$

$$\psi_\nu{}^m(x)=\frac{1}{2i}\,[e^{\mp{}^1/_2 i\pi m}P_{\nu-{}^1/_2}^{-m}(ix)-e^{\pm{}^1/_2 i\pi m}P_{\nu-{}^1/_2}^{-m}(-ix)]\qquad(x\gtrless 0)\qquad(1.4)$$

$P_\nu^{-m}(z)$ is the Legendre function and $\Gamma(z)$ denotes the gamma function. Empty sums appearing in (1.1) when $m = 0$ and in (1.2) when $m = 0$ and $m = 1$ are assumed, as usual, equal to zero, and in these cases the expansions contain only the integral term.

Formulas of this type are of interest for reasons stated at the beginning of this paper, and for a certain narrow class of functions they can be deduced from the general theory of expansions in terms of the characteristic functions [1 and 2].

Our previous paper [3] gives the direct proof of the relations (1.1) and (1.2) for the case $m = 0$. In this paper we attempt to extend the method given in [3] to the general case (*) of arbitrary integral values of m.

The results are given in the form of a theorem.

Theorem. Let $f(x)$ be a given function defined on the interval $(0, \infty)$ and satisfying the following conditions:

1) The function $f(x)$ is piecewise continuous and has a bounded variation in the open interval $(0, \infty)$,

2) The function

$$f(x) \in L(0, a), \qquad f(x)\, x^{-1/2} \ln(1 + x) \in L(a, \infty) \qquad (a > 0)$$

Then $f(x)$ can be represented by the formula (1.1) or (1.2) for any x which is not a discontinuity.

Further, the function $f(x)$ defined on the interval $(-\infty, \infty)$ and satisfying the following conditions:

1) The function $f(x)$ is piecewise continuous and has a bounded variation in the open interval $(-\infty, \infty)$,

2) The function

$$f(x)\ |x|^{-1/2} \ln(1 + |x|) \in L(-\infty, -a)$$

$$f(x)\, x^{-1/2} \ln(1 + x) \in L(a, \infty) \quad (a > 0)$$

can be represented by an analogous formula containing the functions $\phi_\nu{}^m(x)$ and $\psi_\nu{}^m(x)$.

This expression can easily be obtained using Formulas (1.1) and (1.2) in which $f(x)$ is replaced by the following combination of an even and odd function

$$f(x) = {}^1/_2\ [f(x) + f(-x)] + {}^1/_2\ [f(x) - f(-x)]$$

2. Estimates and asymptotic representations of spherical functions. Proof of the expansion theorem is based on certain properties of the functions $\phi_\nu{}^m(x)$ and $\psi_\nu{}^m(x)$, which can be deduced from the expressions giving the spherical functions in terms of the hypergeometric series.

Since the proofs of (1.1) and (1.2) differ from each other only in insignificant details, we shall only consider the even case, using two representations of the function $\phi_\nu{}^m(x)$ following from the definition of the Legendre functions $P_\nu^{-m}(z)$ (see e.g. [5]). the firsy representation is

$$\varphi_\nu{}^m(x) = \frac{\sqrt{\pi}\,(x^2+1)^{-1/2 m}}{2^m \Gamma(3/4 + 1/2 m + 1/2\nu)\,\Gamma(3/4 + 1/2 m - 1/2 \nu)} \times$$

$$\times F\left(\frac{1}{4} - \frac{m}{2} + \frac{\nu}{2},\ \frac{1}{4} - \frac{m}{2} - \frac{\nu}{2},\ \frac{1}{2},\ -x^2\right) \qquad (2.1)$$

and it shows that the function $\phi_\nu{}^m(x)$ is continuous in x over the interval $(0, \infty)$ and, that it is an entire function of the parameter ν.

The second representation is

$$\varphi_\nu{}^m(x) = \frac{\Gamma(3/4 - 1/2 m + 1/2 \nu)\,(x^2+1)^{-1/4}}{2^{m-\nu+1}\Gamma(1+\nu)\,\Gamma(3/4 + 1/2 m - 1/2\nu)} \times$$

*) The case $m = 0$ corresponds to the harmonic boundary value problems for a hyperboloid of one sheet, in which the required function is independent [4] of the polar angle ϕ. When the required function is an arbitrary periodic function of ϕ, we have a general case of integral m, and the method of separation of variables leads to the Sturm-Liouville problem with a mixed spectrum consisting of the interval $(1/4, \infty)$ and a finite number of negative eigenvalues.

$$\times\left\{\left(\sqrt{x^2+1}+x\right)^{\nu}F\left(\frac{1}{2}+m,\ \frac{1}{2}-m,\ 1+\nu,\frac{\sqrt{x^2+1}+x}{2\sqrt{x^2+1}}\right)+\right.$$

$$\left.+\left(\sqrt{x^2+1}-x\right)^{\nu}F\left(\frac{1}{2}+m,\ \frac{1}{2}-m,\ 1+\nu,\ \frac{\sqrt{x^2+1}-x}{2\sqrt{x^2+1}}\right)\right\} \tag{2.2}$$

where $F(a, b, c, x)$ is a hypergeometric function. This formula enables us to obtain the estimates for $\phi_\nu{}^m$, required in the proof of (1.1). Before anything else, we shall note that, when $0 \leq x < 1$ and $\mathrm{Re}\,\nu \geq 0$, we have (*)

$$\left|F\left(\frac{1}{2}+m,\ \frac{1}{2}-m,\ 1+\nu,\ x\right)\right|=\left|\sum_{k=0}^{\infty}\frac{(^1/_2+m)_k\,(^1/_2-m)_k}{(1+\nu)_k\,k!}x^k\right|\leqslant$$

$$\leqslant\sum_{k=0}^{\infty}\frac{|(^1/_2+m)_k\,(^1/_2-m)_k|}{(1)_k\,k!}x^k=O(1)\sum_{k=0}^{\infty}\frac{(^1/_2)_k\,(^1/_2)_k}{(1)_k\,k!}x^k=O(1)\,F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ x\right)$$

which, in turn, gives for $0 < x < \infty$, $0 < \tau < \infty$,

$$\varphi_{i\tau}{}^m(x)=O(1)\,g(x) \tag{2.3}$$

$$g(x)=(x^2+1)^{-1/4}\left\{F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \frac{\sqrt{x^2+1}+x}{2\sqrt{x^2+1}}\right)+F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \frac{\sqrt{x^2+1}-x}{2\sqrt{x^2+1}}\right)\right\}$$

and the estimate is uniform in τ over any interval $(0, T)$.

Considering the behavior of hypergeometric functions as $x \to \infty$ we find, that

$g(x) = O(1)$, $(0 \leqslant x \leqslant a)$, $g(x) = O(1)\,x^{-1/2}\ln(1+x)$ $(a \leqslant x < \infty)$, $(a > 0)$

Next, using the expansion

$$F\left(\frac{1}{2}+m,\ \frac{1}{2}-m,\ 1+\nu,\ x\right)=1+\sum_{k=1}^{\infty}\frac{(^1/_2+m)_k\,(^1/_2-m)_k}{(1+\nu)_k\,k!}x^k=1+r(\nu,\ x) \tag{2.4}$$

and estimating the last term of this formula we find, for $0 < x < 1$ and $\mathrm{Re}\,\nu > 0$,

$$|r(\nu,\ x)|=\left|\sum_{k=1}^{\infty}\frac{(^1/_2+m)_k\,(^1/_2-m)_k}{(1+\nu)_k\,k!}x^k\right|=\left|\frac{^1/_4-m^2}{1+\nu}x\sum_{k=0}^{\infty}\frac{(^3/_2+m)_k\,(^3/_2-m)_k}{(2+\nu)_k\,(k+1)!}x^k\right|\leqslant$$

$$\leqslant\frac{xO(1)}{|1+\nu|}\sum_{k=0}^{\infty}\frac{(^3/_2)_k\,(^3/_2)_k}{(2)_k\,k!}x^k=\frac{xO(1)}{|1+\nu|}F\left(\frac{3}{2},\ \frac{3}{2},\ 2,x\right)$$

hence

$$F\left(\frac{1}{2}+m,\ \frac{1}{2}-m,\ 1+\nu,\ x\right)=1+\frac{x}{1-x}O(|\nu|^{-1})$$

which, together with (2.2), yields the following asymptotic formula

$$\frac{2^{m-\nu}\Gamma(1+\nu)\,\Gamma(^3/_4+{}^1/_2m-{}^1/_2\nu)}{\Gamma(^3/_4-{}^1/_2m+{}^1/_2\nu)}\varphi_\nu{}^m(\mathrm{sh}\,\alpha)= \qquad (|\arg\nu|\leqslant{}^1/_2\pi)$$

$$=\frac{1}{2\sqrt{\mathrm{ch}\,\alpha}}\{e^{\alpha\nu}[1+e^{2\alpha}O(|\nu|^{-1})]+e^{-\alpha\nu}[1+e^{-2\alpha}O(|\nu|^{-1})]\} \qquad \left(\begin{matrix}\alpha\geqslant 0\\ |\nu|\to\infty\end{matrix}\right) \tag{2.5}$$

Let us now consider, in addition to $\phi_\nu{}^m(x)$, another function $\omega_\nu{}^m(x)$ representing a linear combination of Lagrange functions of the second kind $Q_\nu^{-m}(x)$

$$\omega_\nu{}^m(x)=(-1)^m\,{}^1/_2\,[e^{1/_2 i\pi m}Q_{\nu-1/_2}^{-m}(ix)+e^{-1/_2 i\pi m}Q_{\nu-1/_2}^{-m}(-ix)]$$

The latter can be written in terms of the hypergeometric series

$$\omega_\nu{}^m(x)=\frac{\pi\Gamma(^1/_4-{}^1/_2m+{}^1/_2\nu)\,(x^2+1)^{-1/_2\nu-1/_4}}{2^{m+1}\Gamma(1+\nu)\,\Gamma(^1/_4+{}^1/_2m-{}^1/_2\nu)}\times$$

*) Asymptotic behavior of the gamma function implies that $|(\frac{1}{2}+m)_k(\frac{1}{2}-m)_k| = O(1)\,(\frac{1}{2})_k(\frac{1}{2})_k$, where O is independent of k.

$$\times F\left(\frac{1}{4}+\frac{m}{2}+\frac{\nu}{2}, \frac{1}{4}-\frac{m}{2}+\frac{\nu}{2}, 1+\nu, \frac{1}{x^2+1}\right) \tag{2.7}$$

$$\omega_\nu^m(x)=\frac{\pi\Gamma(1/4-1/2m+1/2\nu)(x^2+1)^{-1/4}}{2^{m-\nu+1}\Gamma(1+\nu)\Gamma(1/4+1/2m-1/2\nu)}(\sqrt{x^2+1}+x)^{-\nu}\times$$
$$\times F\left(\frac{1}{2}+m, \frac{1}{2}-m, 1+\nu, \frac{\sqrt{x^2+1}-x}{2\sqrt{x^2+1}}\right) \tag{2.8}$$

In addition, it is continuous over the interval $(0, \infty)$ and is a meromorphic function of ν with poles at the points $\nu = m - 1/2 - 2n$ $(n = 0, 1, 2, \ldots)$.

Finally, from the estimates given above it follows, that this function has the following asymptotic form:

$$\frac{2^{m-\nu}\Gamma(1+\nu)\Gamma(1/4+1/2m-1/2\nu)}{\pi\Gamma(1/4-1/2m+1/2\nu)}\,\omega_\nu^m(\operatorname{sh}\alpha)=\frac{e^{-\alpha\nu}}{2\sqrt{\operatorname{ch}\alpha}}[1+e^{-2\alpha}O(|\nu|)^{-1}]$$
$$\alpha\geqslant 0, \qquad |\nu|\to\infty, \qquad |\arg\nu|\leqslant 1/2\pi \tag{2.9}$$

Formulas (2.3), (2.5) and (2.9) are sufficient to demonstrate the validity of the Eq. (1.1).

3. Proof of the expansion theorem. Let us consider the integral

$$J(T, x)=\frac{2}{\pi}\int_0^T \tau \operatorname{th}\pi\tau\,\Gamma(1/2+m+i\tau)\,\Gamma(1/2+m-i\tau)\,\varphi_{i\tau}^m(x)\,d\tau\int_0^\infty f(y)\,\varphi_{i\tau}^m(y)\,dy$$
$$0<x<\infty, \quad T>0 \tag{3.1}$$

From (2.3) and (2.4) it follows that the inner integral in (3.1) is less than

$$O(1)\int_0^a |f(y)|\,dy+O(1)\int_a^\infty |f(y)|\,y^{-1/2}\ln(1+y)\,dy$$

therefore, by Conditions 2 of the Theorem, it converges absolutely and uniformly on τ in any interval $(0, T)$. Consequently, the expression under the repeated integral in (3.1)(*) is continuous in τ and the integral is valid for any $T > 0$.

Further, the absolute convergence implies, that we can change the order of integration and write $J(T, x)$ in the form

$$J(T, x)=\int_0^\infty f(y)\,K(x, y, T)\,dy \tag{3.2}$$

$$K(x, y, T)=\frac{2}{\pi}\int_0^T \tau\operatorname{th}\pi\tau\,\Gamma(1/2+m+i\tau)\,\Gamma(1/2+m-i\tau)\,\varphi_{i\tau}^m(x)\,\varphi_{i\tau}^m(y)\,d\tau \tag{3.3}$$

or, since the integrand is an even function of τ, then

$$K(x, y, T)=-\frac{1}{\pi i}\int_{-iT}^{iT}\nu\operatorname{tg}\pi\nu\,\Gamma(1/2+m+\nu)\,\Gamma(1/2+m-\nu)\,\varphi_\nu^m(x)\,\varphi_\nu^m(y)\,d\nu \tag{3.4}$$

Now, the well known formula

$$\pi\operatorname{tg}\pi\nu P_{\nu-1/2}^{-m}(z)=Q_{-\nu-1/2}^{-m}(z)-Q_{\nu-1/2}^{-m}(z)$$

yields

$$\pi\operatorname{tg}\pi\nu\varphi_\nu^m(x)=\omega_{-\nu}^m(x)-\omega_\nu^m(x),$$

which can be used to transform (3.4) into one of the following two forms

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*) We note, that the function under the inner integral sign is piecewise continuous on the open interval $(0, \infty)$, and is continuous in τ over the interval $(0, T)$.

$$K(x, y, T) = \frac{2}{\pi^2 i} \int_{-iT}^{iT} \nu\Gamma(1/2 + m + \nu)\, \Gamma(1/2 + m - \nu)\, \omega_\nu{}^m(x)\, \varphi_\nu{}^m(y)\, d\nu \qquad (y \leqslant x)$$

$$K(x, y, T) = \frac{2}{\pi^2 i} \int_{-iT}^{iT} \nu\Gamma(1/2 + m + \nu)\, \Gamma(1/2 + m - \nu)\, \varphi_\nu{}^m(x)\, \omega_\nu{}^m(y)\, d\nu \quad (y \geqslant x) \qquad (3.5)$$

where x is fixed and greater than zero.

Expressions under the integral sign in (3.5) are analytic functions of the complex variable ν, and they have no singularities in the semiplane $\operatorname{Re} \nu \geq 0$, except for the finite number of poles (*)

$$\nu = m - 1/2 - 2n, \quad n = 0, 1, 2, \ldots, [1/2\,(m-1)] \qquad (3.6)$$

Completing the contour of integration on (3.5) with the arc Γ_T of radius $T > m$ situated in the semiplane $\operatorname{Re} \nu \geq 0$ and applying the residue theorem, we obtain

$$K(x, y, T) = \frac{2}{\pi^2 i} \int_{\Gamma_T} \nu\Gamma(1/2 + m + \nu)\, \Gamma(1/2 + m - \nu)\, \omega_\nu{}^m(x)\, \varphi_\nu{}^m(y)\, d\nu - $$
$$- \frac{4}{\pi} \sum_{n=0}^{[1/2(m-1)]} (a_{-1})_{\nu = m - 1/2 - 2n} \qquad (y \leqslant x)$$
$$K(x, y, T) = \frac{2}{\pi^2 i} \int_{\Gamma_T} \nu\Gamma(1/2 + m + \nu)\, \Gamma(1/2 + m - \nu)\, \varphi_\nu{}^m(x)\, \omega_\nu{}^m(y)\, d\nu - $$
$$- \frac{4}{\pi} \sum_{n=0}^{[1/2(m-1)]} (a_{-1})_{\nu = m - 1/2 - 2n} \qquad (y \geqslant x) \qquad (3.7)$$

where

$$(a_{-1})_{\nu = m - 1/2 - 2n} = (m - 1/2 - 2n)\, \Gamma(2m - 2n)\, \Gamma(2n + 1)\, \varphi^m_{m-1/2-2n}(x)\, \varphi^m_{m-1/2-2n}(y) \qquad (3.8)$$

Let us denote by $K_1(x, y, T)$ and $K_2(x, y, T)$ the integrals taken along the contour Γ_T in (3.7). Then (3.2) can be written as (*)

$$J(T, x) = \int_0^x f(y)\, K_1(x, y, T)\, dy + \int_x^\infty f(y)\, K_2(x, y, T)\, dy - \qquad (3.9)$$
$$- \frac{4}{\pi} \sum_{n=0}^{[1/2(m-1)]} (m - 1/2 - 2n)\, \Gamma(2m - 2n)\, \Gamma(2n + 1)\, \varphi^m_{m-1/2-2n}(x) \int_0^\infty f(y)\, \varphi^m_{m-1/2-2n}(y)\, dy$$

Let us now investigate the behavior of $J(T, x)$ as $T \to \infty$. It follows from (2.5) and (2.9), and from the asymptotic formulas for the gamma function that, when $|\nu| \to \infty$ and $|\arg \nu| \leq \leq \frac{1}{2}\pi$, then

$$\nu\pi^{-1}\Gamma(1/2 + m + \nu)\, \Gamma(1/2 + m - \nu)\, \psi_\nu{}^m(\operatorname{sh}\alpha)\, \varphi_\nu{}^m(\operatorname{sh}\alpha') =$$

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*) These poles represent the singularities of the function $\omega_\nu{}^m(x)$ and are situated on the semiplane $\operatorname{Re} \nu \geq 0$. The poles

$$\nu = 1/2 + m + N \quad (N = 0, 1, 2, \ldots)$$

of the function $\Gamma(\frac{1}{2} + m - \nu)$ cancel with the zeros of $\omega_\nu{}^m(x)$ at $N = 2p$ and with the zeros of $\phi_\nu{}^m(x)$ at $N = 2p + 1$ ($p = 0, 1, 2, \ldots$) (see Formulas (2.1) and (2.7)).

**) We note that the definition of $\phi_\nu{}^m(x)$ readily yields

$$\varphi^m_{m-1/2-2n}(x) = O(1) \quad (0 \leqslant x \leqslant 1), \qquad \varphi^m_{m-1/2-2n}(x) = O(1)\, x^{-m+2n}, \quad (x \geqslant 1)$$

it follows that the integrals under the summation sign converge absolutely.

$$= \frac{1}{4\sqrt{\operatorname{ch}\alpha \operatorname{ch}\alpha'}}\{e^{-(\alpha-\alpha')\nu} + e^{-(\alpha+\alpha')\nu} + e^{-(\alpha-\alpha')\nu}O(|\nu|^{-1}) + e^{-(\alpha+\alpha')\nu}O(|\nu|^{-1})\} \qquad (0 \leqslant \alpha' \leqslant \alpha)$$

$$\nu\pi^{-1}\Gamma(1/2 + m + \nu)\,\Gamma(1/2 + m - \nu)\,\psi_\nu^m(\operatorname{sh}\alpha')\,\varphi_\nu^m(\operatorname{sh}\alpha) =$$

$$= \frac{1}{4\sqrt{\operatorname{ch}\alpha \operatorname{ch}\alpha'}}\{e^{-(\alpha'-\alpha)\nu} + e^{-(\alpha'+\alpha)\nu} + e^{-(\alpha'-\alpha)\nu}O(|\nu|^{-1}) + e^{-(\alpha'+\alpha)\nu}O(|\nu|^{-1})\} \qquad (\alpha \leqslant \alpha' < \infty \qquad (3.10)$$

Assuming now that

$$\nu = Te^{i\varphi}\;(-1/2\,\pi \leqslant \varphi \leqslant 1/2\pi)$$

holds along the arc Γ_T and using (3.10), we find that

$$K_1(\operatorname{sh}\alpha, \operatorname{sh}\alpha', T) = \frac{1}{\sqrt{\operatorname{ch}\alpha \operatorname{ch}\alpha'}}\left\{\frac{\sin(\alpha-\alpha')T}{\pi(\alpha-\alpha')} + \frac{\sin(\alpha+\alpha')T}{\pi(\alpha+\alpha')} + O(1)\frac{1-e^{-(\alpha-\alpha')T}}{(\alpha-\alpha')T} + O(1)\frac{1-e^{-(\alpha+\alpha')T}}{(\alpha+\alpha')T}\right\} \qquad (0 \leqslant \alpha' \leqslant \alpha) \qquad (3.11)$$

$$K_2(\operatorname{sh}\alpha, \operatorname{sh}\alpha', T) = \frac{1}{\sqrt{\operatorname{ch}\alpha \operatorname{ch}\alpha'}}\left\{\frac{\sin(\alpha'-\alpha)T}{\pi(\alpha'-\alpha)} + \frac{\sin(\alpha'+\alpha)T}{\pi(\alpha'+\alpha)} + O(1)\frac{1-e^{-(\alpha'-\alpha)T}}{(\alpha'-\alpha)T} + O(1)\frac{1-e^{-(\alpha'+\alpha)T}}{(\alpha'+\alpha)T}\right\} \qquad (\alpha \leqslant \alpha' < \infty)$$

Subsequent arguments are analogous to those given in [3] and are, therefore, omitted.

Let us now put $x = \operatorname{sh}\alpha$ and $y = \operatorname{sh}\alpha'$ in (3.9); using (3.11) we can represent the integrals taken over the intervals $(0, x)$ and (x, ∞) as sums of other integrals. For example,

$$\int_0^x f(y)\,K_1(x, y, T)\,dy = \int_0^\alpha f(\operatorname{sh}\alpha')\,K_1(\operatorname{sh}\alpha, \operatorname{sh}\alpha', T)\operatorname{ch}\alpha'\,d\alpha' =$$

$$= \frac{1}{\pi}\int_0^\alpha f(\operatorname{sh}\alpha')\left(\frac{\operatorname{ch}\alpha'}{\operatorname{ch}\alpha}\right)^{1/2}\frac{\sin(\alpha-\alpha')T}{\alpha-\alpha'}\,d\alpha' + \frac{1}{\pi}\int_0^\alpha f(\operatorname{sh}\alpha')\left(\frac{\operatorname{ch}\alpha'}{\operatorname{ch}\alpha}\right)^{1/2}\frac{\sin(\alpha+\alpha')T}{\alpha+\alpha'}\,d\alpha' +$$

$$+ O(1)\int_0^\alpha |f(\operatorname{sh}\alpha')|\left(\frac{\operatorname{ch}\alpha'}{\operatorname{ch}\alpha}\right)^{1/2}\frac{1-e^{-(\alpha-\alpha')T}}{(\alpha-\alpha')T}\,d\alpha' + O(1)\int_0^\alpha |f(\operatorname{sh}\alpha')|\left(\frac{\operatorname{ch}\alpha'}{\operatorname{ch}\alpha}\right)^{1/2}\frac{1-e^{-(\alpha+\alpha')T}}{(\alpha+\alpha')T}\,d\alpha'$$

From the conditions imposed on the function $f(x)$ we find, in accordance with the Dirichlet theorem,

$$\lim_{T\to\infty}\frac{1}{\pi}\int_0^\alpha f(\operatorname{sh}\alpha')\left(\frac{\operatorname{ch}\alpha'}{\operatorname{ch}\alpha}\right)^{1/2}\frac{\sin(\alpha-\alpha')T}{\alpha-\alpha'}\,d\alpha' = \frac{1}{2}f(\operatorname{sh}\alpha - 0)$$

The remaining integrals tend to zero with increasing T (we must, however, adopt a certain procedure when the integral in question has the difference $\alpha - \alpha'$ appearing in the denominator; a δ-neighborhood of α must be taken and T increased without bounds, with δ kept sufficiently small).

$$\lim_{T\to\infty}\int_0^x f(y)\,K_1(x,\ y,\ T)\,dy=\frac{1}{2}\,f(x-0) \tag{3.12}$$

Similarly, we can show that

$$\lim_{T\to\infty}\int_x^\infty f(y)\,K_2(x,\ y,\ T)\,dy=\frac{1}{2}\,f(x+0) \tag{3.13}$$

Thus

$$\lim_{T\to\infty} J(T,\ x)={}^1\!/_2\,[f(x+0)+f(x-0)]- \tag{3.14}$$

$$-\frac{4}{\pi}\sum_{n=0}^{[{}^1\!/_2(m-1)]}(m-{}^1\!/_2-2n)\,\Gamma(2m-2n)\,\Gamma(2n+1)\,\varphi^m_{m-{}^1\!/_2-2n}(x)\int_0^\infty f(y)\,\varphi^m_{m-{}^1\!/_2-2n}(y)\,dy.$$

which proves the validity of (1.1).

4. Examples. Let us consider the expansion of $f(x)=(x^2+1)^{-{}^1\!/_2 s}$ $(s>{}^1\!/_2)$. Conditions of the theorem will be fulfilled, and the expansion (1.1) will therefore exist. Computation of the integrals over the variable y can be performed by replacing $\phi_\nu{}^m(y)$ with the relevant expression given by (2.2). The required expansion has the form

$$(x^2+1)^{-{}^1\!/_2 s}=\frac{2^{m-1}}{\pi\Gamma({}^1\!/_2 s+{}^1\!/_2 m)\,\Gamma({}^1\!/_2 s-{}^1\!/_2 m)}\times$$

$$\times\Big\{2\sum_{n=0}^{[{}^1\!/_2(m-1)]}\Big(m-\frac{1}{2}-2n\Big)\,\Gamma\Big(n+\frac{1}{2}\Big)\,\Gamma(m-n)\,\Gamma\Big(\frac{s}{2}-\frac{1}{2}+\frac{m}{2}-n\Big)\times$$

$$\times\,\Gamma\Big(\frac{s}{2}-\frac{m}{2}+n\Big)\,\varphi^m_{m-{}^1\!/_2-2n}(x)+\int_0^\infty\tau\,\mathrm{th}\,\pi\tau\,\Gamma\Big(\frac{1}{4}+\frac{m}{2}+\frac{i\tau}{2}\Big)\Gamma\Big(\frac{1}{4}+\frac{m}{2}-\frac{i\tau}{2}\Big)\times$$

$$\times\,\Gamma\Big(\frac{s}{2}-\frac{1}{4}+\frac{i\tau}{2}\Big)\Gamma\Big(\frac{s}{2}-\frac{1}{4}-\frac{i\tau}{2}\Big)\,\varphi^m_{i\tau}(x)\,d\tau\Big\}$$

$$0\leqslant x<\infty,\qquad s>{}^1\!/_2,\qquad m=0,1,2,\ldots \tag{4.1}$$

A derivation of the potential of a point source in terms of the eigenfunctions of the boundary value harmonic problem for a hyperboloid of revolution of one sheet, can serve as another example of application of the above theorems. We omit, however, the final expression in view of its complexity.

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